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# On Laplacian of Quotient of Randić and Sum-Connectivity Energy of Graphs

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**Abstract**

In this paper we define the Laplacian of quotient of Randić and sum-connectivity energy of a graph.Then we compute the Laplacian of quotient of Randić and sum-connectivity energies of complete graph, star graph, complete bipartite graph, the Crown graph, the $(S_m \wedge P_2)$ graph.

**Key words:** Laplacian of quotient of randić and sum-connectivity energy

**2010 Mathematics Subject Classification** : 05C50

## 1 Introduction

In [6] we define quotient of Randić and sum-connectivity energy of a simple graph $G$ as follows. Let $a$ and $b$ be two nonnegative real number with $a \neq 0$. The quotient of Randić and sum-connectivity adjacency matrix of $G$ is the $n \times n$ matrix $A_{qrs} = (a_{ij})$ where

$$a_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{\frac{a(d_i+d_j)}{b(d_id_j)}}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

The eigenvalues of the graph $G$ are the eigenvalues of $A_{qrs}$. Since $A_{qrs}$ is real and symmetric, its eigenvalues are real numbers which are denoted by $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$, where $\lambda_1 \geq \lambda_2 \geq$

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$\lambda_3 \geq ... \geq \lambda_n$. Then the Quotient of Randić and sum-connectivity energy of $G$ is defined as

$$E_{qrs}(G) = \sum_{i=1}^{n} \mid \lambda_i \mid .$$

In 2004, D. Vukičević and Gutman [5] have defined the Laplacian matrix of the graph $G$, denoted by $L = (L_{ij})$, as a square matrix of order $n$ whose elements are defined by

$$L_{ij} = \begin{cases} \delta_i, & \text{if } i = j, \\ -1, & \text{if } i \neq j \text{ and the vertices } v_i, v_j \text{ are adjacent,} \\ 0, & \text{if } i \neq j \text{ and the vertices } v_i, v_j \text{ are not adjacent,} \end{cases}$$

where $\delta_i$ is the degree of vertex $v_i$. The eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of $L$, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are called the Laplacian eigenvalues of $G$. In 2006, Gutman and B. Zhou [2] have defined the Laplacian energy of $LE(G)$ of $G$ as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$$

where $m$ is number of edges and $n$ is number of vertices of $G$.

Motivated by these works, we introduce the Laplacian of Quotient of Randić and sum-connectivity energy of a simple graph $G$ as follows. Let $a$ and $b$ be two nonnegative real number with $a \neq 0$. The Laplacian of quotient of Randić and sum-connectivity adjacency matrix of $G$ is the $n \times n$ matrix $A_{lqrs} = (a_{ij})$ where

$$a_{ij} = \begin{cases} \delta_i, & \text{if } i = j, \\ \frac{1}{\sqrt{\frac{a(d_i+d_j)}{b(d_i d_j)}}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent.} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

where $\delta_i$ is the degree of vertex $v_i$. Where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are called the eigenvalues of $A_{lqrs}$. Then Laplacian of Quotient of Randić and sum-connectivity energy of $G$ is

$$E_{lqrs}(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$$

where $m$ is number of edges and $n$ is number of vertices of $G$.

## 2 Laplacian of Quotient of Randić and sum-connectivity energies of some families of graphs

We begin with some basic definitions and notations.

**Definition 2.1.** [3] A graph $G$ is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on $n$ vertices is denoted by $K_n$.

**Definition 2.2.** [1] The Crown graph $S_n^0$ for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edge set $\{u_i v_j; 1 \leq i, j \leq n, i \neq j\}$. $S_n^0$ is therefore equivalent to the complete bipartite graph $K_{n,n}$ from which the edges of perfect matching have been removed.

**Definition 2.3.** [3] A bigraph or bipartite graph $G$ is a graph whose vertex set $V(G)$ can be partitioned into two subsets $V_1$ and $V_2$ such that every line of $G$ joins $V_1$ with $V_2$. $(V_1, V_2)$ is a bipartition of $G$. If $G$ contains every line joining $V_1$ and $V_2$, then $G$ is a complete bigraph. If $V_1$ and $V_2$ have $m$ and $n$ points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$.

**Definition 2.4.** [4] The conjunction $(S_m \wedge P_2)$ of $S_m = \overline{K}_m + K_1$ and $P_2$ is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l) | v_i v_k \in E(S_m) \ and \ v_j v_l \in E(P_2) \ and \ 1 \leq i, k \leq m+1, 1 \leq j, l \leq 2\}$.

Now we compute Laplacian of Quotient of Randić and sum-connectivity energies of complete graph, star graph, complete bipartite graph, the Crown graph, the $(S_m \wedge P_2)$ graph.

**Theorem 2.5.** Let a and b be as defined above. Then the Laplacian of Quotient of Randić and sum-connectivity energy of the complete bipartite graph $K_{n,n}$ is $2\sqrt{\frac{b(n^2)^2}{a(n+n)}}$.

**Proof:** Let the vertex set of the complete bipartite graph be $V(K_{n,n}) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$. Then the Laplacian of Quotient of Randić and sum-connectivity matrix of complete bipartite graph is given by

$$A_{lqrs} = \begin{pmatrix} n & \cdots & 0 & \sqrt{\frac{b(n^2)}{a(n+n)}} & \cdots & \sqrt{\frac{b(n^2)}{a(n+n)}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & n & \sqrt{\frac{b(n^2)}{a(n+n)}} & \cdots & \sqrt{\frac{b(n^2)}{a(n+n)}} \\ \sqrt{\frac{b(n^2)}{a(n+n)}} & \cdots & \sqrt{\frac{b(n^2)}{a(n+n)}} & n & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{b(n^2)}{a(n+n)}} & \cdots & \sqrt{\frac{b(n^2)}{a(n+n)}} & & \cdots & n \end{pmatrix}.$$

Its characteristic polynomial is

$$|\lambda I - A_{lqrs}| = \begin{vmatrix} (\lambda - n)I_n & -\sqrt{\frac{b(n^2)}{a(n+n)}}J^T \\ -\sqrt{\frac{b(n^2)}{a(n+n)}}J & (\lambda - n)I_n \end{vmatrix}$$

where $J$ is an $n \times n$ matrix with all the entries are equal to 1. Hence the characteristic equation is given by

$$\begin{vmatrix} \Lambda I_n & -\sqrt{\frac{b(n^2)}{a(n+n)}}J^T \\ -\sqrt{\frac{b(n^2)}{a(n+n)}}J & (\Lambda - n)I_n \end{vmatrix} = 0$$

where $\Lambda = \lambda - n$ and which can be written as

$$|\Lambda I_n| \left| \Lambda I_n - \left(-\sqrt{\frac{b(n^2)}{a(n+n)}}J\right) \frac{I_n}{\Lambda} \left(-\sqrt{\frac{b(n^2)}{a(n+n)}}J^T\right) \right| = 0.$$

On simplification, we obtain

$$\frac{\Lambda^{n-n}}{(\frac{(a(n+n)}{b(n^2)})^n} \left| \frac{a(n+n)}{b(n^2)}\Lambda^2 I_n - JJ^T \right| = 0$$

which can be written as

$$\frac{\Lambda^{n-n}}{(\frac{a(n+n)}{b(n^2)})^n} P_{JJ^T}(\frac{a(n+n)}{b(n^2)})\Lambda^2) = 0$$

where $P_{JJ^T}(\lambda)$ is the characteristic polynomial of the matrix ${}_nJ_n$. Thus, we have

$$\frac{\Lambda^{n-n}}{(\frac{a(n+n)}{bn^2})^n}((\frac{a(n+n)}{b(n^2)})\Lambda^2 - n^2)(\frac{a(n+n)}{b(n^2)}\Lambda^2)^{n-1} = 0$$

which is same as

$$\Lambda^{n+n-2}(\Lambda^2 - \frac{b(n^2)^2}{a(n+n)}) = 0.$$

Therefore, the spectrum of $K_{n,n}$ is given by

$$Spec\,(K_{n,n}) = \begin{pmatrix} n & n + \sqrt{\frac{b(n^2)^2}{a(n+n)}} & n - \sqrt{\frac{b(n^2)^2}{a(n+n)}} \\ n+n-2 & 1 & 1 \end{pmatrix}.$$

Hence the Laplacian of Quotient of Randić and sum-connectivity energy of the complete bipartite graph is

$$E_{lqrs}(K_{n,n}) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$$

$E_{lqrs}(K_{n,n}) = 2\sqrt{\frac{b(n^2)^2}{a(n+n)}}$ as desired. ■

**Theorem 2.6.** Let a and b be as defined above. Then the Laplacian of quotient of Randić and sum-connectivity energy of the $S_n$ is

$$E_{lqrs}(S_n) = \frac{(n-2)^2}{n} + \left|\frac{n^2-2(n-1)}{n} + \sqrt{\frac{n^3a - 4(n-1)(an - b(n-1))}{2na}}\right| + \left|\frac{n^2-2(n-1)}{n} - \sqrt{\frac{n^3a - 4(n-1)(an - b(n-1))}{2na}}\right|.$$

**Proof:** Let the vertex set of star graph be given by $V(S_n) = \{v_1, v_2, \ldots, v_n\}$. Then the Laplacian of quotient of Randić and sum-connectivity matrix of the star graph $S_n$ is given by

$$A_{lqrs} = \begin{pmatrix} n-1 & \sqrt{\frac{b(n-1)}{an}} & \sqrt{\frac{b(n-1)}{an}} & \cdots & \sqrt{\frac{b(n-1)}{an}} & \sqrt{\frac{b(n-1)}{an}} \\ \sqrt{\frac{b(n-1)}{an}} & 1 & 0 & \cdots & 0 & 0 \\ \sqrt{\frac{b(n-1)}{an}} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{\frac{b(n-1)}{an}} & 0 & 0 & \cdots & 1 & 0 \\ \sqrt{\frac{b(n-1)}{an}} & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Hence the characteristic polynomial is given by

$$|\lambda I - A_{lqrs}| = \begin{vmatrix} \lambda-(n-1) & -\sqrt{\frac{b(n-1)}{an}} & -\sqrt{\frac{b(n-1)}{an}} & \cdots & -\sqrt{\frac{b(n-1)}{an}} \\ -\sqrt{\frac{b(n-1)}{an}} & \lambda-1 & 0 & \cdots & 0 \\ -\sqrt{\frac{b(n-1)}{an}} & 0 & \lambda-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{b(n-1)}{an}} & 0 & 0 & \cdots & \lambda-1 \end{vmatrix}$$

$$= \left(\sqrt{\frac{b(n-1)}{an}}\right)^n \begin{vmatrix} \gamma & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & 0 & \cdots & 0 & 0 \\ -1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & \mu & 0 \\ -1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix},$$

where $\mu = (\lambda - 1)\sqrt{\frac{an}{b(n-1)}}$ and $\gamma = (\lambda - (n-1))\sqrt{\frac{an}{b(n-1)}}$. Then $|\lambda I - A_{lqrs}| = \phi_n(\mu)\left(\sqrt{\frac{b(n-1)}{a(n)}}\right)^n$

where $\phi_n(\mu) = \begin{vmatrix} \gamma & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & 0 & \cdots & 0 & 0 \\ -1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & \mu & 0 \\ -1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}$

Using the properties of the determinants, we obtain after some simplifications

$$\phi_n(\mu) = (\mu\phi_{n-1}(\mu) - \mu^{n-2})$$

Iterating this, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu\gamma - (n-1))$$

Therefore

$$|\lambda I - A_{lqrs}| = \left(\sqrt{\frac{b(n-1)}{an}}\right)^n \left[\left((\frac{an}{b(n-1)})(\lambda-1)(\lambda-(n-1)) - (n-1)\right)((\lambda-1)\sqrt{\frac{b(n-1)}{an}})^{n-2}\right].$$

Thus the characteristic equation is given by

$$(\lambda-1)^{n-2}\left((\lambda-1)(\lambda-(n-1)) - \frac{b(n-1)^2}{an}\right) = 0$$

Hence

$$Spec\,(S_n) = \begin{pmatrix} 1 & n + \sqrt{\frac{n^3a-4(n-1)(an-b(n-1))}{2na}} & n - \sqrt{\frac{n^3a-4(n-1)(an-b(n-1))}{2na}} \\ n-2 & 1 & 1 \end{pmatrix}$$

Hence the Laplacian of quotient of Randić and sum-connectivity energy of $S_n$ is

$$E_{lqrs}(S_n) = \frac{(n-2)^2}{n} + \left|\frac{n^2-2(n-1)}{n} + \sqrt{\frac{n^3a-4(n-1)(an-b(n-1)}{2na}}\right| + \left|\frac{n^2-2(n-1)}{n} - \sqrt{\frac{n^3a-4(n-1)(an-b(n-1)}{2na}}\right|.$$

■

**Theorem 2.7.** Let a and b be as defined above. Then the Laplacian of quotient of Randić and sum-connectivity energy of $K_n$ is $2(n-1)\sqrt{\frac{(n-1)b}{a2}}$ .

**Proof:** Let the vertex set of Complete graph be given by $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Then the Laplacian of quotient of Randić and sum-connectivity energy of matrix of the complete graph

$K_n$ is given by

$$A_{lqrs} = \begin{pmatrix} n-1 & \sqrt{\frac{(n-1)^2b}{a2(n-1)}} & \cdots & \sqrt{\frac{(n-1)^2b}{a2(n-1)}} \\ \sqrt{\frac{(n-1)^2b}{a2(n-1)}} & n-1 & \cdots & \sqrt{\frac{(n-1)^2b}{a2(n-1)}} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{(n-1)^2b}{a2(n-1)}} & \sqrt{\frac{(n-1)^2b}{a2(n-1)}} & \cdots & n-1 \end{pmatrix}$$

Hence the characteristic polynomial is given by

$$|\lambda I - A_{lqrs}| = \begin{vmatrix} \lambda-(n-1) & -\sqrt{\frac{(n-1)^2b}{a2(n-1)}} & \cdots & -\sqrt{\frac{(n-1)^2b}{a2(n-1)}} \\ -\sqrt{\frac{(n-1)^2b}{a2(n-1)}} & \lambda-(n-1) & \cdots & -\sqrt{\frac{(n-1)^2b}{a2(n-1)}} \\ \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{(n-1)^2b}{a2(n-1)}} & -\sqrt{\frac{(n-1)^2b}{a2(n-1)}} & \cdots & \lambda-(n-1) \end{vmatrix}$$

$$= \left(\sqrt{\frac{(n-1)^2b}{a2(n-1)}}\right)^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix}$$

where $\mu = (\lambda - (n-1))\sqrt{\frac{a2(n-1)}{(n-1)^2b}}$. Then $|\lambda I - A_{lqrs}| = \phi_n(\mu)\left(\sqrt{\frac{(n-1)^2b}{a2(n-1)}}\right)^n$,

where $\phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix}$

$$= \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ 0 & 0 & 0 & \cdots & -1-\mu & \mu+1 \end{vmatrix}$$

$$= (\mu+1)\begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix} + (\mu+1)\begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix}.$$

$$\phi_n(\mu) = -(\mu+1)^{n-1} + (\mu+1)\left[(\mu+1)^{n-2}(\mu-(n-2))\right]$$
$$= -(\mu+1)^{n-1} + (\mu+1)^{n-1}(\mu-(n-2)).$$

Iterating this, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu^2-(n-1)),$$

Thus the characteristic equation is given by

$$\left(\sqrt{\frac{(n-1)^2 b}{a2(n-1)}}\right)^n (\mu+1)^{n-1}(\mu-(n-1)) = 0.$$

Hence the Laplacian of Quotient of Randić and sum-connectivity energy of $K_n$ is

$$E_{lqrs}(K_n) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|$$

$$E_{lqrs}(K_n) = 2(n-1)\sqrt{\frac{(n-1)b}{a2}}.$$

∎

**Theorem 2.8.** Let the vertex set $V(S_n^0)$ of the crown graph be given by

$$V(S_n^0) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}.$$

Then the Laplacian of quotient of Randić and sum-connectivity energy of the crown graph is$4(n-1)\sqrt{\frac{(n-1)b}{a2}}$.

**Proof:** The Laplacian of quotient of Randić and sum-connectivity energy matrix of crown graph is given by

$$A_{lqrs} = \begin{pmatrix} n-1 & 0 & \cdots & 0 & 0 & \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & \cdots & \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} \\ 0 & n-1 & \cdots & 0 & \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & 0 & \cdots & \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n-1 & \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & \cdots & 0 \\ 0 & \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & \cdots & \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & n-1 & 0 & \cdots & 0 \\ \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & 0 & \cdots & \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & 0 & n-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & \cdots & 0 & 0 & 0 & \cdots & n-1 \end{pmatrix}.$$

Its characteristic polynomial is

$$|\lambda I - A_{lqrs}| = \begin{vmatrix} (\lambda-(n-1))I_n & -\sqrt{\frac{b((n-1)^2)}{a(2(n-1))}}K^T \\ \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}}K & (\lambda-(n-1)I_n \end{vmatrix}$$

where $K$ is an $n \times n$ matrix. Hence the characteristic equation is given by

$$\begin{vmatrix} \Lambda I_n & -\sqrt{\frac{b((n-1)^2)}{a(2(n-1))}}K^T \\ \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}}K & \Lambda I_n. \end{vmatrix} = 0.$$

Where $\Lambda = (\lambda-(n-1)$, this is same as

$$|\Lambda I_n|\left|\Lambda I_n - \left(-\sqrt{\frac{b((n-1)^2)}{a(2(n-1))}}K\right)\frac{I_n}{\Lambda}\left(-\sqrt{\frac{b((n-1)^2)}{a(2(n-1))}}K^T\right)\right| = 0$$

which can be written as

$$\left(\frac{b((n-1)^2)}{a(2(n-1))}\right)^n P_{KK^T}\left(\left(\frac{a(2(n-1))}{b((n-1)^2)}\right)\Lambda^2\right) = 0$$

where $P_{KK^T(\Lambda)}$ is the characteristic polynomial of the matrix $KK^T$. Thus we have

$$\left(\frac{b((n-1)^2)}{a(2(n-1))}\right)^n [\frac{a(2(n-1))}{b((n-1)^2)})\Lambda^2 - (n-1)^2][\frac{a(2(n-1))}{b((n-1)^2)})\Lambda^2 - 1]^{n-1} = 0$$

which is same as
$$\left(\Lambda^2 - \frac{b(n-1)^3}{2a}\right)\left(\Lambda^2 - \frac{b(n-1)}{a2}\right)^{n-1} = 0.$$

Therefore

$$Spec\,(S_n^0) = \begin{pmatrix} \sqrt{\frac{b(n-1)^3}{2a}} + (n-1) & -\sqrt{\frac{b(n-1)^3}{2a}} + (n-1) & \sqrt{\frac{b(n-1)}{2a}} + (n-1) & -\sqrt{\frac{b(n-1)}{2a}} + (n-1) \\ 1 & 1 & n-1 & n-1 \end{pmatrix}.$$

Hence the Laplacian of quotient of Randić and sum-connectivity energy of crown graph is $E_{lqrs}(S_n^0) = 4(n-1)\sqrt{\frac{(n-1)b}{a2}}$ as desired. ∎

**Theorem 2.9.** Let a and b be as defined above. Then the Laplacian of Quotient of Randić and sum-connectivity energy of $(S_m \wedge P_2)$ is

$$\frac{(2n-4)(2-n)}{n} + 2\left|\frac{n^2-2(n-1)}{n} + \sqrt{\frac{n^3a-4(n-1)(an-b(n-1))}{2na}}\right| + 2\left|\frac{n^2-2(n-1)}{n} - \sqrt{\frac{n^3a-4(n-1)(an-b(n-1))}{2na}}\right|$$

**Proof:** Let the vertex set of $(S_m \wedge P_2)$ graph be given by $V(S_m \wedge P_2) = \{v_1, v_2, \ldots, v_{2m+2}\}$. Then the Laplacian of Quotient of Randić and sum-connectivity matrix of $(S_m \wedge P_2)$ graph is given by

$$A_{lqrs} = \begin{pmatrix} n-1 & 0 & \cdots & 0 & 0 & \sqrt{\frac{b(n-1)}{an}} & \cdots & \sqrt{\frac{b(n-1)}{an}} \\ 0 & 1 & \cdots & 0 & \sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ 0 & \sqrt{\frac{b(n-1)}{an}} & \cdots & \sqrt{\frac{b(n-1)}{an}} & n-1 & 0 & \cdots & 0 \\ \sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \end{pmatrix}_{2n\times 2n}$$

where $m+1=n$. Its characteristic polynomial is given by

$$|\lambda I - A_{lqrs}| = \begin{vmatrix} \lambda-(n-1) & 0 & \cdots & 0 & 0 & -\sqrt{\frac{b(n-1)}{an}} & \cdots & -\sqrt{\frac{b(n-1)}{an}} \\ 0 & \lambda-1 & \cdots & 0 & -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda-1 & -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ 0 & -\sqrt{\frac{b(n-1)}{an}} & \cdots & -\sqrt{\frac{b(n-1)}{an}} & \lambda-(n-1) & 0 & \cdots & 0 \\ -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 & 0 & \lambda-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda-1 \end{vmatrix}_{2n\times 2n}.$$

Hence the characteristic equation is given by

$$\left(\sqrt{\frac{b(n-1)}{an}}\right)^{2n}\begin{vmatrix}\gamma & 0 & \cdots & 0 & 0 & -1 & \cdots & -1\\ 0 & \Lambda & \cdots & 0 & -1 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0\\ 0 & -1 & \cdots & -1 & \gamma & 0 & \cdots & 0\\ -1 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda\end{vmatrix}_{2n\times 2n} = 0$$

$$where\ \Lambda = \sqrt{\frac{na}{(n-1)b}}(\lambda-1) and \gamma = \sqrt{\frac{na}{(n-1)b}}(\lambda-(n-1)).$$

Let

$$\phi_{2n}(\Lambda) = \begin{vmatrix}\gamma & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1\\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0\\ 0 & 0 & \Lambda & \cdots & 0 & -1 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & 0 & \cdots & 0\\ 0 & -1 & -1 & \cdots & -1 & \gamma & 0 & 0 & \cdots & 0\\ -1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda\end{vmatrix}_{2n\times 2n}$$

$$= (-1)^{2n+2n}\Lambda\begin{vmatrix}\gamma & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1\\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0\\ 0 & 0 & \Lambda & \cdots & 0 & -1 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & 0 & \cdots & 0\\ 0 & -1 & -1 & \cdots & -1 & \gamma & 0 & 0 & \cdots & 0\\ -1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda\end{vmatrix}_{(2n-1)\times(2n-1)}$$

$$+(-1)^{2n+2}\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \gamma & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}$$

Let

$$\Psi_{2n-1}(\Lambda) = (-1)^{2n+2}\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \gamma & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}$$

Using the properties of the determinants, we obtain, after some simplifications

$$\Psi_{2n-1}(\Lambda) = -\Lambda^{n-2}\Theta_n(\Lambda)$$

where $\Theta_n(\Lambda)=\begin{vmatrix} \Lambda & 0 & 0 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & -1 \\ 0 & 0 & \Lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \gamma \end{vmatrix}_{n\times n}$

Then

$$\phi_{2n}(\Lambda) = -\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda\phi_{2n-1}(\Lambda).$$

Now, proceeding as above, we obtain

$$\begin{aligned} \phi_{2n-1}(\Lambda) &= (-1)^{(2n-1)+2}\Psi_{2n-2}(\Lambda) + (-1)^{(2n-1)+(2n-1)}\Lambda\phi_{2n-2}(\Lambda) \\ &= -\Lambda^{n-3}\Theta_n(\Lambda) + \Lambda\phi_{2n-2}(\Lambda). \end{aligned}$$

Proceeding like this, we obtain at the $(n-1)^{th}$ step

$$\phi_{2n}(\Lambda) = -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{(n-1)}\xi_{n+1}(\Lambda)$$

where $\xi_{n+1}(\Lambda) = \begin{vmatrix} \gamma & 0 & 0 & \cdots & 0 \\ 0 & \Lambda & 0 & \cdots & -1 \\ 0 & 0 & \Lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{(n+1)\times(n+1)}$

$$\begin{aligned} \phi_{2n}(\Lambda) &= -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\gamma\Theta_n(\Lambda) \\ &= -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\gamma\Theta_n(\Lambda) \\ &= (\Lambda^{n-1}\gamma - (n-1)\Lambda^{n-2})\Theta_n(\Lambda). \end{aligned}$$

Using the properties of the determinants, we obtain

$$\Theta_n(\Lambda) = \Lambda^{n-1}\gamma - (n-1)\Lambda^{n-2}$$

Therefore

$$\phi_{2n}(\Lambda) = (\Lambda^{n-1}\gamma - (n-1)\Lambda^{n-2})^2$$

Hence characteristic equation becomes

$$\left(\sqrt{\frac{b(n-1)}{an}}\right)^{2n} \phi_{2n}(\Lambda) = 0$$

which is same as

$$\left(\sqrt{\frac{b(n-1)}{an}}\right)^{2n} (\Lambda^{n-1}\gamma - (n-1)\Lambda^{n-2})^2 = 0$$

This reduces to

$$\lambda^{2n-4}((\frac{na}{b(n-1)}(\lambda-1)(\lambda-(n-1)) - (n-1))^2 = 0.$$

Therefore

$$Spec\left((S_m \wedge P_2)\right) = \begin{pmatrix} 1 & n + \sqrt{\frac{n^3a-4(n-1)(an-b(n-1))}{2na}} & n - \sqrt{\frac{n^3a-4(n-1)(an-b(n-1))}{2na}} \\ 2n-4 & 2 & 2 \end{pmatrix}$$

Hence the Laplacian of Quotient of Randić and sum-connectivity energy of $(S_m \wedge P_2)$ graph is

$$E_{lqrs}((S_m \wedge P_2)) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$$

$$E_{lqrs}((S_m \wedge P_2)) = \frac{(2n-4)(2-n)}{n} + 2\left| \frac{n^2-2(n-1)}{n} + \sqrt{\frac{n^3a-4(n-1)(an-b(n-1))}{2na}} \right| + 2\left| \frac{n^2-2(n-1)}{n} - \sqrt{\frac{n^3a-4(n-1)(an-b(n-1)))}{2na}} \right|.$$

■

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