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Sum Divisor Cordial Labeling of Bipartite, Fan, and Related Graphs

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Abstract

A sum divisor cordial labeling of a graph G with vertex set $V(G)$ is a bijection f from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$ such that an edge uv is assigned the label 1 if $f(u) + f(v)$ is even and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph that admits a sum divisor cordial labeling is called a sum divisor cordial graph. In this work, we establish that the complete bipartite graph $K_{m,n}$, the splitting graph $S'(K_{m,n})$, the bi-usual fan graph $B(F_{1,n})$, and the fan graph $F_{m,n}$ all admit sum divisor cordial labelings for every natural number m and n.

Key words: Sum divisor cordial labeling, Sum divisor cordial graph.

2020 Mathematics Subject Classification : 05C78

1 Introduction and Preliminaries

The concept of sum divisor cordial labeling was introduced by Lourdusamy and Patrick [7], where they initiated the study of sum divisor cordial labelings of various graphs, including graph operations. Subsequently, several researchers extended this study to other families of graphs; for further details, the reader may consult [1, 3, 4, 5, 10, 11]. In addition, Gallian [6] provides a dynamic survey of graph labeling, which also summarizes previous results on sum divisor cordial labeling.

In the present work, we build upon certain results from [2, 7, 8], which established that the double fan $F_{2,n}$, the star $K_{1,n}$, the graphs $K_{2,n}$ and $K_2 + mK_1$, as well as the splitting graph of the star $S'(K_{1,n})$, are all sum divisor cordial graphs. We extend these results to the complete bipartite graph $K_{m,n}$, its splitting graph $S'(K_{m,n})$, the fan graph $F_{m,n}$, and additionally study the bi-usual fan graph $B(F_{1,n})$. For each of these graphs, we construct explicit sum divisor cordial labelings covering all natural numbers m and n.

We begin by recalling the definitions and notation used throughout this work, including those pertaining to the graphs under consideration. Throughout the paper, all graphs are assumed to be finite, simple, nontrivial, and connected. For a graph G with vertex set

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$V(G)$ and edge set $E(G)$, we write $|V(G)|$ and $|E(G)|$ for the numbers of vertices and edges of G, respectively.

Throughout this paper, the symbol $\mathbb{N}$ denotes the set of natural numbers.

Definition 1.1 ([2])**.** Let $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ be a bijection and let $uv \in E(G)$. Consider the induced function $f^* : E(G) \to \{0, 1\}$ defined by

$$f^*(uv) = \begin{cases} 1, & \text{if } f(u) + f(v) \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

The bijection f is called a sum divisor cordial labeling of G if $|e_f(0) - e_f(1)| \le 1$, where $e_f(i)$ denotes the number of edges assigned label i. A graph which admits a sum divisor cordial labeling is called a sum divisor cordial graph.

Definition 1.2. ([8]) For a graph G, the splitting graph $S'(G)$ of a graph G is obtained from G by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.

Definition 1.3 ([9])**.** A fan graph $F_{m,n}$ is the graph join $\overline{K}_m + P_n$, where $\overline{K}_m$ denotes the complement of K_m. For $m = 1$ and $m = 2$, this construction yields the usual fan graph and the double fan graph, respectively.

Definition 1.4. A bi-usual fan $B(F_{1,n})$ is the graph obtained by attaching the apex vertices of two copies of $F_{1,n}$ by an edge.

To facilitate our proofs, we make use of several important results that have been established in previous works. The following lemmas summarize the known sum divisor cordial graphs that will be used in this study.

Lemma 1.5. *([2]) The double fan $F_{2,n}$ is a sum divisor cordial graph for all $n \in \mathbb{N}$.*

Lemma 1.6. *([7]) The star $K_{1,n}$ is a sum divisor cordial graph for all $n \in \mathbb{N}$.*

Lemma 1.7. *([7]) The graph $K_{2,n}$ is a sum divisor cordial graph for all $n \in \mathbb{N}$.*

Lemma 1.8. *([7]) The graph $K_2 + mK_1$ is a sum divisor cordial graph for all $m \in \mathbb{N}$.*

Lemma 1.9. *([8]) The splitting graph of the star $S'(K_{1,n})$ is a sum divisor cordial graph for all $n \in \mathbb{N}$.*

2 Main Results

Theorem 2.1. *The complete bipartite graph $K_{m,n}$ is a sum divisor cordial graph for all $m, n \in \mathbb{N}$.*

Proof: Let $V(K_{m,n}) = X \cup Y$, where $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ are the partite sets. Then $|V(K_{m,n})| = m + n$ and $|E(K_{m,n})| = mn$.

Observe that when $m = 1$ or $n = 1$, the graph $K_{m,n}$ is either $K_{1,n}$ or $K_{m,1}$, both of which are sum divisor cordial graphs by Lemma 1.6.

Therefore, it suffices to consider the case where $m \geq 2$ and $n \geq 2$. Define a bijection $f : V(K_{m,n}) \to \{1, 2, \ldots, m + n - 1, m + n\}$ by setting $f(x_i) = i$ for $i = 1, 2, \ldots, m$ and $f(y_j) = m + j$ for $j = 1, 2, \ldots, n$.
One can verify that the edge counts $e_f(1)$ and $e_f(0)$ satisfy the following with respect to m and n:

$$e_f(1) = \begin{cases} \frac{mn-1}{2}, & \text{if both } m \text{ and } n \text{ are odd,} \\ \frac{mn}{2}, & \text{otherwise,} \end{cases} \qquad e_f(0) = \begin{cases} \frac{mn+1}{2}, & \text{if both } m \text{ and } n \text{ are odd,} \\ \frac{mn}{2}, & \text{otherwise.} \end{cases}$$

Hence, $|e_f(0) - e_f(1)| \leq 1$. ■

Example 2.2. Figure 1 illustrates a sum divisor cordial labeling of $K_{2,3}$, where solid edges represent edges labeled 1 and dashed edges represent edges labeled 0.

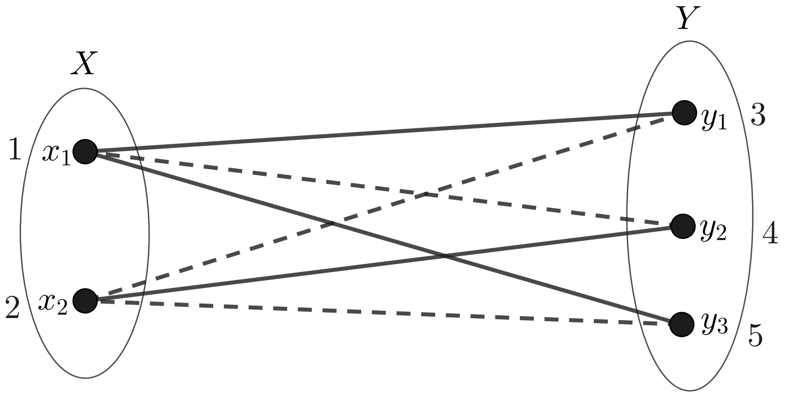

Figure 1: A sum divisor cordial labeling of $K_{2,3}$

Theorem 2.3. *The bi-usual fan graph $B(F_{1,n})$ is a sum divisor cordial graph for all $n \in \mathbb{N}$.*

Proof: Let $V(B(F_{1,n})) = \{x_0, x_1, x_2, \ldots, x_n\} \cup \{y_0, y_1, y_2, \ldots, y_n\}$, where $x_0, y_0 \in V(\overline{K}_1)$ and x_0, y_0 are apex vertices. Then $|V(B(F_{1,n}))| = 2n + 2$ and $|E(B(F_{1,n}))| = 4n - 1$.

For the small cases $n = 1, 2, 3$, we define bijections $f : V(B(F_{1,n})) \to \{1, 2, \ldots, 2n + 1, 2n + 2\}$ directly.

For $n = 1$, set $f(x_1) = 1, f(x_0) = 3, f(y_0) = 2, f(y_1) = 4$.

For $n = 2$, set $f(x_1) = 2, f(x_0) = 4, f(y_0) = 6, f(x_2) = 5, f(y_1) = 1, f(y_2) = 3$.

For $n = 3$, set $f(x_1) = 5, f(x_2) = 7, f(y_1) = 1, f(y_2) = 3, f(x_3) = 2, f(x_0) = 4, f(y_0) = 6, f(y_3) = 8$.

It is routine to check that each of these bijections yields a sum divisor cordial labeling.

We now consider the case $n \geq 4$. Write $n = 4t + r$, where $t \in \mathbb{N}$ and $0 \leq r \leq 3$. Define a bijection $f : V(B(F_{1,n})) \to \{1, 2, \ldots, 2n+1, 2n+2\}$ by setting $f(x_0) = 1$ and $f(y_0) = 2$. For the remaining vertices $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, define f according to the remainder r in the following cases.

Case 1. $r = 0$.

$$f(x_i) = \begin{cases} 4k-1, & \text{if } i = 4k-3,\ 1 \leq k \leq \frac{n}{4}, \\ 4k+1, & \text{if } i = 4k-2,\ 1 \leq k \leq \frac{n}{4}, \\ 4k, & \text{if } i = 4k-1,\ 1 \leq k \leq \frac{n}{4}, \\ 4k+2, & \text{if } i = 4k,\ 1 \leq k \leq \frac{n}{4}, \end{cases}$$

$$f(y_j) = \begin{cases} n+2+(4k-3), & \text{if } j = 4k-3,\ 1 \leq k \leq \frac{n}{4}, \\ n+2+(4k-1), & \text{if } j = 4k-2,\ 1 \leq k \leq \frac{n}{4}, \\ n+2+(4k-2), & \text{if } j = 4k-1,\ 1 \leq k \leq \frac{n}{4}, \\ n+2+4k, & \text{if } j = 4k,\ 1 \leq k \leq \frac{n}{4}. \end{cases}$$

Case 2. $r = 1$.

$$f(x_i) = \begin{cases} 4k-1, & \text{if } i = 4k-3,\ 1 \leq k \leq \frac{n-1}{4}, \\ 4k+1, & \text{if } i = 4k-2,\ 1 \leq k \leq \frac{n-1}{4}, \\ 4k, & \text{if } i = 4k-1,\ 1 \leq k \leq \frac{n-1}{4}, \\ 4k+2, & \text{if } i = 4k,\ 1 \leq k \leq \frac{n-1}{4}, \\ 2n+2, & \text{if } i = n, \end{cases}$$

$$f(y_j) = \begin{cases} n+2+(4k-2), & \text{if } j = 4k-3,\ 1 \leq k \leq \frac{n-1}{4}, \\ n+2+4k, & \text{if } j = 4k-2,\ 1 \leq k \leq \frac{n-1}{4}, \\ n+2+(4k-3), & \text{if } j = 4k-1,\ 1 \leq k \leq \frac{n-1}{4}, \\ n+2+(4k-1), & \text{if } j = 4k,\ 1 \leq k \leq \frac{n-1}{4}, \\ n+2, & \text{if } j = n. \end{cases}$$

Case 3. $r = 2$.

$$f(x_i) = \begin{cases} 4k-1, & \text{if } i = 4k-3,\ 1 \le k \le \frac{n-2}{4}, \\ 4k+1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{n-2}{4}, \\ 4k, & \text{if } i = 4k-1,\ 1 \le k \le \frac{n-2}{4}, \\ 4k+2, & \text{if } i = 4k,\ 1 \le k \le \frac{n-2}{4}, \\ n+1, & \text{if } i = n-1, \\ n+2, & \text{if } i = n, \end{cases}$$

$$f(y_j) = \begin{cases} n+2+(4k-2), & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-2}{4}, \\ n+2+4k, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-2}{4}, \\ n+2+(4k-3), & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-2}{4}, \\ n+2+(4k-1), & \text{if } j = 4k,\ \ 1 \le k \le \frac{n-2}{4}, \\ 2n+1, & \text{if } j = n-1, \\ 2n+2, & \text{if } j = n. \end{cases}$$

Case 4. $r = 3$.

$$f(x_i) = \begin{cases} 4k-1, & \text{if } i = 4k-3,\ 1 \le k \le \frac{n-3}{4}, \\ 4k+1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{n-3}{4}, \\ 4k, & \text{if } i = 4k-1,\ 1 \le k \le \frac{n-3}{4}, \\ 4k+2, & \text{if } i = 4k,\ 1 \le k \le \frac{n-3}{4}, \\ n, & \text{if } i = n-2, \\ n+1, & \text{if } i = n-1, \\ n+2, & \text{if } i = n, \end{cases}$$

$$f(y_j) = \begin{cases} n+2+(4k-2), & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-3}{4}, \\ n+2+4k, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-3}{4}, \\ n+2+(4k-3), & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-3}{4}, \\ n+2+(4k-1), & \text{if } j = 4k,\ \ 1 \le k \le \frac{n-3}{4}, \\ 2n, & \text{if } j = n-2, \\ 2n+1, & \text{if } j = n-1, \\ 2n+2, & \text{if } j = n. \end{cases}$$

It is straightforward to verify that $e_f(1) = 2n$ and $e_f(0) = 2n-1$ in Case 1, while $e_f(1) = 2n-1$ and $e_f(0) = 2n$ in all remaining cases. Thus, $|e_f(0) - e_f(1)| \le 1$. ■

Example 2.4. Figure 2 illustrates a sum divisor cordial labeling of $B(F_{1,4})$, where solid edges represent edges labeled 1 and dashed edges represent edges labeled 0.

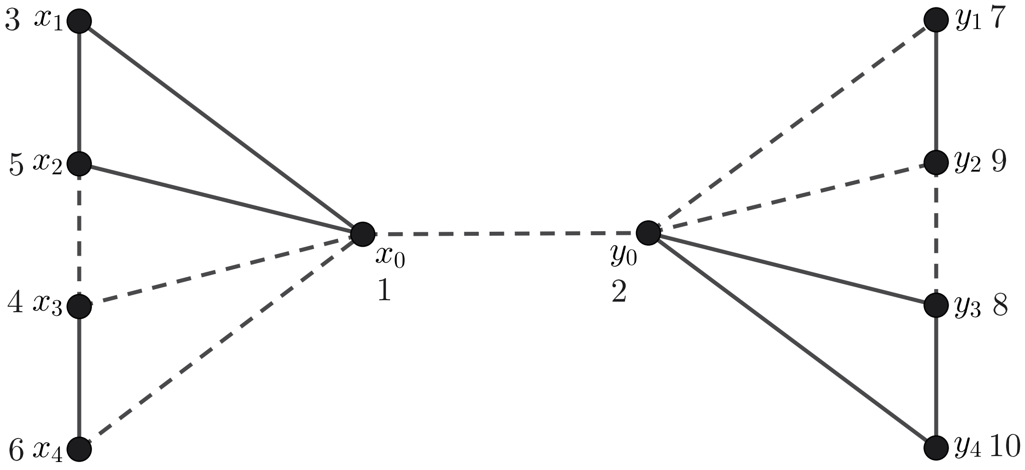

Figure 2: A sum divisor cordial labeling of $B(F_{1,4})$

Lemma 2.5. *The usual fan graph $F_{1,n}$ is a sum divisor cordial graph for all $n \in \mathbb{N}$.*

Proof: Let $V(F_{1,n}) = \{x_1\} \cup \{y_1, y_2, \ldots, y_n\}$, where $x_1 \in V(\overline{K}_1)$ and $y_1, y_2, \ldots, y_n \in V(P_n)$. Then $|V(F_{1,n})| = n+1$ and $|E(F_{1,n})| = 2n-1$.
For the small cases $n = 1, 2, 3$, we define bijections $f : V(F_{1,n}) \to \{1, 2, \ldots, n, n+1\}$ directly.

For $n = 1$, let $f(x_1) = 1$ and $f(y_1) = 2$.

For $n = 2$, set $f(x_1) = 1$, $f(y_1) = 2$, and $f(y_2) = 3$.

For $n = 3$, set $f(x_1) = 1$, $f(y_1) = 2$, $f(y_2) = 4$, and $f(y_3) = 3$.

Each of these bijections can be verified to be a sum divisor cordial labeling.

We now proceed to the case $n \geq 4$. Let $n = 4t + r$, where $t \in \mathbb{N}$ and $0 \leq r \leq 3$. Define a bijection $f : V(F_{1,n}) \to \{1, 2, \ldots, n, n+1\}$ by setting $f(x_1) = n+1$ and assigning labels to the remaining vertices $y_1, y_2, \ldots, y_n$ according to the remainder r in the following cases.

Case 1. $r = 0$. Define $f(y_j) = \begin{cases} 4k-3, & \text{if } j = 4k-3,\ 1 \leq k \leq \frac{n}{4}, \\ 4k-1, & \text{if } j = 4k-2,\ 1 \leq k \leq \frac{n}{4}, \\ 4k-2, & \text{if } j = 4k-1,\ 1 \leq k \leq \frac{n}{4}, \\ 4k, & \text{if } j = 4k,\ 1 \leq k \leq \frac{n}{4}. \end{cases}$

Case 2. $r = 1$. Define $f(y_j) = \begin{cases} 4k-3, & \text{if } j = 4k-3,\ 1 \leq k \leq \frac{n-1}{4}, \\ 4k-1, & \text{if } j = 4k-2,\ 1 \leq k \leq \frac{n-1}{4}, \\ 4k-2, & \text{if } j = 4k-1,\ 1 \leq k \leq \frac{n-1}{4}, \\ 4k, & \text{if } j = 4k,\ 1 \leq k \leq \frac{n-1}{4}, \\ n, & \text{if } j = n. \end{cases}$

Case 3. $r = 2$. Define $f(y_j) = \begin{cases} 4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-2}{4}, \\ 4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-2}{4}, \\ 4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-2}{4}, \\ 4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-2}{4}, \\ n-1, & \text{if } j = n-1, \\ n, & \text{if } j = n. \end{cases}$

Case 4. $r = 3$. Define $f(y_j) = \begin{cases} 4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-3}{4}, \\ 4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-3}{4}, \\ 4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-3}{4}, \\ 4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-3}{4}, \\ n-2, & \text{if } j = n-2, \\ n, & \text{if } j = n-1, \\ n-1, & \text{if } j = n. \end{cases}$

It is straightforward to verify that $e_f(1) = n$ and $e_f(0) = n-1$ in Case 1, while $e_f(1) = n-1$ and $e_f(0) = n$ in all remaining cases. Thus, $|e_f(0) - e_f(1)| \le 1$. ■

Theorem 2.6. *The fan graph $F_{m,n}$ is a sum divisor cordial graph for all $m, n \in \mathbb{N}$.*

Proof: Let $V(\overline{K}_m) = \{x_1, x_2, \ldots, x_m\}$ and $V(P_n) = \{y_1, y_2, \ldots, y_n\}$. Then $|V(F_{m,n})| = m + n$ and $|E(F_{m,n})| = mn + n - 1$.

If $m = 1$, then $F_{m,n}$ is $F_{1,n}$, which is known to be a sum divisor cordial graph by Lemma 2.5.

If $n = 1$, then $F_{m,n}$ is $K_{m,1}$, which is known to be a sum divisor cordial graph by Lemma 1.6.

If $m = 2$ or $n = 2$, then $F_{m,n}$ is either $F_{2,n}$ or $F_{m,2}$. The graph $F_{2,n}$ is a sum divisor cordial graph by Lemma 1.5, whereas $F_{m,2}$ is isomorphic to $K_2 + mK_1$ and is a sum divisor cordial graph by Lemma 1.8. Hence, we may assume that $m \ge 3$ and $n \ge 3$, and we divide the proof into two parts.

In the first part, we consider Cases 1 through 3.

Case 1. Let $m = n = 3$. Define a bijection $f : V(F_{3,3}) \to \{1, 2, 3, 4, 5, 6\}$ by $f(x_1) = 1, f(x_2) = 2, f(x_3) = 4$, and $f(y_1) = 3, f(y_2) = 5, f(y_3) = 6$. One easily checks that f is a sum divisor cordial labeling.

Case 2. Let $m = 3$ and $n = 4t + r$, where $t \in \mathbb{N}$ and $0 \le r \le 3$. Define a bijection $f : V(F_{m,n}) \to \{1, 2, \ldots, n+2, n+3\}$ according to the remainder r in the following subcases.

Subcase 2.1. $r = 0$.

For the vertices x_1, x_2, x_3, define $f(x_1) = 1, f(x_2) = 3$, and $f(x_3) = 2$.

For the vertices $y_1, y_2, \ldots, y_n$, define $f(y_j) = \begin{cases} 4k+1, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n}{4}, \\ 4k+3, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n}{4}, \\ 4k+2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n}{4}, \\ 4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n}{4}. \end{cases}$

Subcase 2.2. $r = 1$.

For the vertices x_1, x_2, x_3, define $f(x_1) = n+1, f(x_2) = n+2$, and $f(x_3) = n+3$.

For the vertices $y_1, y_2, \ldots, y_n$, define $f(y_j) = \begin{cases} 4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-1}{4}, \\ 4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-1}{4}, \\ 4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-1}{4}, \\ 4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-1}{4}, \\ n, & \text{if } j = n. \end{cases}$

Subcase 2.3. $r = 2$.

For the vertices x_1, x_2, x_3, define $f(x_1) = 1, f(x_2) = 2$, and $f(x_3) = 3$.

For the vertices $y_1, y_2, \ldots, y_n$, define $f(y_j) = \begin{cases} 4k+3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-2}{4}, \\ 4k+1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-2}{4}, \\ 4k, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-2}{4}, \\ 4k+2, & \text{if } j = 4k, \quad 1 \le k \le \frac{n-2}{4}. \\ n+3, & \text{if } j = n-1, \\ n+2, & \text{if } j = n. \end{cases}$

Subcase 2.4. $r = 3$.

For the vertices x_1, x_2, x_3, define $f(x_1) = n+1, f(x_2) = n+2$, and $f(x_3) = n+3$.

For the vertices $y_1, y_2, \ldots, y_n$, define $f(y_j) = \begin{cases} 4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-3}{4}, \\ 4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-3}{4}, \\ 4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-3}{4}, \\ 4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-3}{4}. \\ n-2, & \text{if } j = n-2, \\ n, & \text{if } j = n-1, \\ n-1, & \text{if } j = n. \end{cases}$

In Subcase 2.1, one checks that $e_f(1) = 2n$ and $e_f(0) = 2n-1$, whereas in all other subcases we have $e_f(1) = 2n-1$ and $e_f(0) = 2n$.

Case 3. Let $m = 4t + r$, where $t \in \mathbb{N}$ and $0 \le r \le 3$, and let $n = 3$. Define a bijection $f : V(F_{m,n}) \to \{1, 2, \ldots, m+2, m+3\}$ according to the remainder r in the following subcases.

Subcase 3.1. $r = 0$.

For the vertices $x_1, x_2, \ldots, x_m$, define $f(x_i) = \begin{cases} 4k-3, & \text{if } i = 4k-3,\ 1 \le k \le \frac{m}{4}, \\ 4k-1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{m}{4}, \\ 4k-2, & \text{if } i = 4k-1,\ 1 \le k \le \frac{m}{4}, \\ 4k, & \text{if } i = 4k,\ 1 \le k \le \frac{m}{4}. \end{cases}$

For the vertices y_1, y_2, y_3, define $f(y_1) = m+1, f(y_2) = m+3$, and $f(y_3) = m+2$.

It is straightforward to verify that $e_f(1) = e_f(0) = \frac{3m}{2} + 1$.

Subcase 3.2. $r = 1$.

For the vertices $x_1, x_2, \ldots, x_m$, define $f(x_i) = \begin{cases} 4k-3, & \text{if } i = 4k-3,\ 1 \le k \le \frac{m-1}{4}, \\ 4k-1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{m-1}{4}, \\ 4k-2, & \text{if } i = 4k-1,\ 1 \le k \le \frac{m-1}{4}, \\ 4k, & \text{if } i = 4k,\ 1 \le k \le \frac{m-1}{4}, \\ m+1, & \text{if } i = m. \end{cases}$

For the vertices y_1, y_2, y_3, define $f(y_1) = m, f(y_2) = m+2$, and $f(y_3) = m+3$.
It is straightforward to verify that

$$e_f(1) = \frac{3(m-1)}{2} + 2, \quad e_f(0) = \frac{3(m-1)}{2} + 3.$$

Subcase 3.3. $r = 2$.

For the vertices $x_1, x_2, \ldots, x_m$, define $f(x_i) = \begin{cases} 4k+3, & \text{if } i = 4k-3,\ 1 \le k \le \frac{m-2}{4}, \\ 4k+1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{m-2}{4}, \\ 4k, & \text{if } i = 4k-1,\ 1 \le k \le \frac{m-2}{4}, \\ 4k+2, & \text{if } i = 4k,\ 1 \le k \le \frac{m-2}{4}, \\ m+3, & \text{if } i = m-1, \\ m+2, & \text{if } i = m. \end{cases}$

For the vertices y_1, y_2, y_3, define $f(y_1) = 1, f(y_2) = 3$, and $f(y_3) = 2$.
It is straightforward to verify that $e_f(1) = e_f(0) = \frac{3(m-2)}{2} + 4$.

Subcase 3.4. $r = 3$.

For the vertices $x_1, x_2, \ldots, x_{m-3}$, define $f(x_i) = \begin{cases} 4k+3, & \text{if } i = 4k-3,\ 1 \le k \le \frac{m-3}{4}, \\ 4k+1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{m-3}{4}, \\ 4k, & \text{if } i = 4k-1,\ 1 \le k \le \frac{m-3}{4}, \\ 4k+2, & \text{if } i = 4k,\ 1 \le k \le \frac{m-3}{4}, \\ m+1, & \text{if } i = m-2, \\ m+2, & \text{if } i = m-1, \\ m+3, & \text{if } i = m. \end{cases}$

For the vertices y_1, y_2, y_3, define $f(y_1) = 1, f(y_2) = 3$, and $f(y_3) = 2$.
It is straightforward to verify that

$$e_f(1) = \frac{3(m-3)}{2} + 5, \quad e_f(0) = \frac{3(m-3)}{2} + 6.$$

In the second part, we consider Cases 4 through 7, which deal with the case $m, n \ge 4$.

Case 4. Let $m = 4t_1$ and $n = 4t_2 + r$, where $t_1, t_2 \in \mathbb{N}$ and $0 \le r \le 3$.
Define a bijection $f : V(F_{m,n}) \to \{1, 2, \ldots, m+n-1, m+n\}$ as follows.
For the vertices $x_1, x_2, \ldots, x_m$, define f as in Subcase 3.1. For the vertices $y_1, y_2, \ldots, y_n$, define f according to the remainder r in the following subcases.

Subcase 4.1. $r = 0$.

Define $f(y_j) = \begin{cases} m+4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n}{4}, \\ m+4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n}{4}, \\ m+4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n}{4}, \\ m+4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n}{4}. \end{cases}$

It is straightforward to verify that

$$e_f(1) = \frac{mn+n}{2}, \quad e_f(0) = \frac{mn+n}{2} - 1.$$

Subcase 4.2. $r = 1$. Define

$$f(y_j) = \begin{cases} m+4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-1}{4}, \\ m+4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-1}{4}, \\ m+4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-1}{4}, \\ m+4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-1}{4}, \\ m+n, & \text{if } j = n. \end{cases}$$

It is straightforward to verify that $e_f(1) = e_f(0) = \frac{mn+n-1}{2}$.

Subcase 4.3. $r = 2$.

Define $f(y_j) = \begin{cases} m+4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-2}{4}, \\ m+4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-2}{4}, \\ m+4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-2}{4}, \\ m+4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-2}{4}, \\ m+n-1, & \text{if } j = n-1, \\ m+n, & \text{if } j = n. \end{cases}$

It is straightforward to verify that

$$e_f(1) = \frac{mn+n}{2} - 1, \quad e_f(0) = \frac{mn+n}{2}.$$

Subcase 4.4. $r = 3$.

Define

$$f(y_j) = \begin{cases} m+4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-3}{4}, \\ m+4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-3}{4}, \\ m+4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-3}{4}, \\ m+4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-3}{4}, \\ m+n-2, & \text{if } j = n-2, \\ m+n, & \text{if } j = n-1, \\ m+n-1, & \text{if } j = n. \end{cases}$$

It is straightforward to verify that $e_f(1) = e_f(0) = \frac{mn+n-1}{2}$.

Case 5. Let $m = 4t_1 + 1$ and $n = 4t_2 + r$, where $t_1, t_2 \in \mathbb{N}$ and $0 \le r \le 3$. Define a bijection $f : V(F_{m,n}) \to \{1, 2, \dots, m+n-1, m+n\}$ according to the remainder r as the following subcases.

Subcase 5.1. $r = 0$.

Define

$$f(x_i) = \begin{cases} 4k-3, & \text{if } i = 4k-3,\ 1 \le k \le \frac{m-1}{4}, \\ 4k-1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{m-1}{4}, \\ 4k-2, & \text{if } i = 4k-1,\ 1 \le k \le \frac{m-1}{4}, \\ 4k, & \text{if } i = 4k,\ 1 \le k \le \frac{m-1}{4}, \\ m, & \text{if } i = m, \end{cases}$$

$$f(y_j) = \begin{cases} m+4k-2, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n}{4}, \\ m+4k, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n}{4}, \\ m+4k-3, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n}{4}, \\ m+4k-1, & \text{if } j = 4k,\ 1 \le k \le \frac{n}{4}. \end{cases}$$

Subcase 5.2. $r = 1$.

Define

$$f(x_i) = \begin{cases} 4k-3, & \text{if } i = 4k-3,\ 1 \le k \le \frac{m-1}{4}, \\ 4k-1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{m-1}{4}, \\ 4k-2, & \text{if } i = 4k-1,\ 1 \le k \le \frac{m-1}{4}, \\ 4k, & \text{if } i = 4k,\ 1 \le k \le \frac{m-1}{4}, \\ m+n, & \text{if } i = m, \end{cases}$$

$$f(y_j) = \begin{cases} m+4k-4, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-1}{4}, \\ m+4k-2, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-1}{4}, \\ m+4k-3, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-1}{4}, \\ m+4k-1, & \text{if } j = 4k,\ 1 \le k \le \frac{n-1}{4}, \\ m+n-1, & \text{if } i = n. \end{cases}$$

Subcase 5.3. $r = 2$.

For the vertices $x_1, x_2, \dots, x_m$, define f as in Subcase 5.1.

For the vertices $y_1, y_2, \dots, y_n$, define $f(y_j) = \begin{cases} m+4k-2, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-2}{4}, \\ m+4k, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-2}{4}, \\ m+4k-3, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-2}{4}, \\ m+4k-1, & \text{if } j = 4k,\ 1 \le k \le \frac{n-2}{4}, \\ m+n, & \text{if } j = n-1, \\ m+n-1, & \text{if } j = n. \end{cases}$

Subcase 5.4. $r = 3$.

For the vertices $x_1, x_2, \dots, x_m$, define f as in Subcase 5.2.

For the vertices $y_1, y_2, \ldots, y_n$, define $f(y_j) = \begin{cases} m+4k-4, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-3}{4}, \\ m+4k-2, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-3}{4}, \\ m+4k-3, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-3}{4}, \\ m+4k-1, & \text{if } j = 4k,\ 1 \le k \le \frac{n-3}{4}, \\ m+n-3, & \text{if } j = n-2, \\ m+n-1, & \text{if } j = n-1, \\ m+n-2, & \text{if } j = n. \end{cases}$

In Subcase 5.1, it can be checked that $e_f(1) = \frac{mn+n}{2}$ and $e_f(0) = \frac{mn+n}{2} - 1$, whereas in all other subcases, $e_f(1) = \frac{mn+n}{2} - 1$ and $e_f(0) = \frac{mn+n}{2}$.

Case 6. Let $m = 4t_1 + 2$ and $n = 4t_2 + r$, where $t_1, t_2 \in \mathbb{N}$ and $0 \le r \le 3$.
Define a bijection $f : V(F_{m,n}) \to \{1, 2, \ldots, m+n-1, m+n\}$ as follows.

For the vertices $x_1, x_2, \ldots, x_m$, define $f(x_i) = \begin{cases} 4k-3, & \text{if } i = 4k-3,\ 1 \le k \le \frac{m-2}{4}, \\ 4k-1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{m-2}{4}, \\ 4k-2, & \text{if } i = 4k-1,\ 1 \le k \le \frac{m-2}{4}, \\ 4k, & \text{if } i = 4k,\ 1 \le k \le \frac{m-2}{4}, \\ m-1, & \text{if } i = m-1, \\ m, & \text{if } i = m. \end{cases}$

For the vertices $y_1, y_2, \ldots, y_n$, define f according to the remainder r in the following subcases.

Subcase 6.1. $r = 0$.
Define f as in Subcase 4.1.
It is straightforward to verify that $e_f(1) = \frac{mn+n}{2}, \quad e_f(0) = \frac{mn+n}{2} - 1$.

Subcase 6.2. $r = 1$. Define f as in Subcase 4.2.
It is straightforward to verify that $e_f(1) = e_f(0) = \frac{mn+n-1}{2}$.

Subcase 6.3. $r = 2$.
Define f as in Subcase 4.3.
It is straightforward to verify that $e_f(1) = \frac{mn+n}{2} - 1, \quad e_f(0) = \frac{mn+n}{2}$.

Subcase 6.4. $r = 3$.
Define f as in Subcase 4.4. It is straightforward to verify that $e_f(1) = e_f(0) = \frac{mn+n-1}{2}$.

Case 7. Let $m = 4t_1 + 3$ and $n = 4t_2 + r$, where $t_1, t_2 \in \mathbb{N}$ and $0 \le r \le 3$.
Define a bijection $f : V(F_{m,n}) \to \{1, 2, \ldots, m+n-1, m+n\}$ according to the remainder r as the following subcases.

Subcase 7.1. $r = 0$.

For the vertices $x_1, x_2, \ldots, x_m$, define $f(x_i) = \begin{cases} 4k-3, & \text{if } i = 4k-3,\ 1 \le k \le \frac{m-3}{4}, \\ 4k-1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{m-3}{4}, \\ 4k-2, & \text{if } i = 4k-1,\ 1 \le k \le \frac{m-3}{4}, \\ 4k, & \text{if } i = 4k,\ 1 \le k \le \frac{m-3}{4}, \\ m-2, & \text{if } i = m-2, \\ m-1, & \text{if } i = m-1, \\ m, & \text{if } i = m. \end{cases}$

For the vertices $y_1, y_2, \ldots, y_n$, define f as in Subcase 5.1.

Subcase 7.2. $r = 1$. Define

$$f(x_i) = \begin{cases} 4k-3, & \text{if } i = 4k-3,\ 1 \le k \le \frac{m-3}{4}, \\ 4k-1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{m-3}{4}, \\ 4k-2, & \text{if } i = 4k-1,\ 1 \le k \le \frac{m-3}{4}, \\ 4k, & \text{if } i = 4k,\ 1 \le k \le \frac{m-3}{4}, \\ m+n-2, & \text{if } i = m-2, \\ m+n-1, & \text{if } i = m-1, \\ m+n, & \text{if } i = m, \end{cases}$$

$$f(y_j) = \begin{cases} m+4k-6, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-1}{4}, \\ m+4k-4, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-1}{4}, \\ m+4k-5, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-1}{4}, \\ m+4k-3, & \text{if } j = 4k,\ 1 \le k \le \frac{n-1}{4}, \\ m+n-3, & \text{if } j = n. \end{cases}$$

Subcase 7.3. $r = 2$.

For the vertices $x_1, x_2, \ldots, x_m$, define f as in Subcase 7.1.

For the vertices $y_1, y_2, \ldots, y_n$, define f as in Subcase 5.3.

Subcase 7.4. $r = 3$.

For the vertices $x_1, x_2, \ldots, x_m$, define f as in Subcase 7.2.

For the vertices $y_1, y_2, \ldots, y_n$,

$$f(y_j) = \begin{cases} m+4k-6, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-3}{4}, \\ m+4k-4, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-3}{4}, \\ m+4k-5, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-3}{4}, \\ m+4k-3, & \text{if } j = 4k,\ 1 \le k \le \frac{n-3}{4}, \\ m+n-5, & \text{if } j = n-2, \\ m+n-3, & \text{if } j = n-1, \\ m+n-4, & \text{if } j = n. \end{cases}$$

In Subcase 7.1, one has $e_f(1) = \frac{mn+n}{2}$ and $e_f(0) = \frac{mn+n}{2} - 1$, whereas in all other subcases,

$e_f(1) = \frac{mn+n}{2} - 1$ and $e_f(0) = \frac{mn+n}{2}$.
From all above cases, we obtain that $|e_f(0) - e_f(1)| \leq 1$. ■

Example 2.7. Figure 3 illustrates a sum divisor cordial labeling of $F_{4,4}$, where solid edges represent edges labeled 1 and dashed edges represent edges labeled 0.

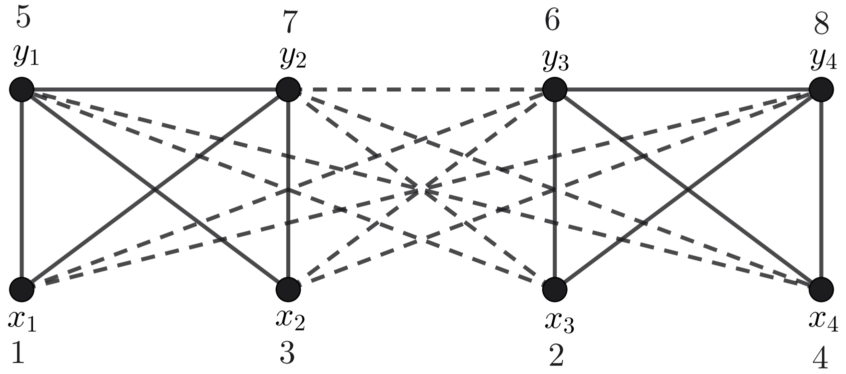

Figure 3: A sum divisor cordial labeling of $F_{4,4}$

Theorem 2.8. *The graph $S'(K_{m,n})$ is a sum divisor cordial graph for all $m, n \in \mathbb{N}$.*

Proof: Let $V(K_{m,n}) = X \cup Y$ with $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ being its partite sets. For $1 \leq i \leq m$ and $1 \leq j \leq n$, we let $X' = \{x'_i : 1 \leq i \leq m\}$ and $Y' = \{y'_j : 1 \leq j \leq n\}$ where x'_i and y'_j are added vertices corresponding to the vertices x_i and y_j, respectively to obtain $S'(K_{m,n})$. Then $|V(S'(K_{m,n}))| = 2m + 2n$ and $|E(S'(K_{m,n}))| = 3mn$. Observe that when $m = 1$ or $n = 1$, the graph $S'(K_{m,n})$ is either $S'(K_{1,n})$ or $S'(K_{m,1})$, both of which are sum divisor cordial graphs by Lemma 1.9.

Therefore, it suffices to consider the case where $m \geq 2$ and $n \geq 2$. Define a bijection $f : V(S'(K_{m,n})) \to \{1, 2, \ldots, 2m + 2n - 1, 2m + 2n\}$, with its explicit form depending on the parity of m and n in the following cases.

Case 1. m and n are even.

$$f(x_i) = \begin{cases} 2i, & \text{if } 1 \leq i \leq \frac{m}{2}, \\ 2\left(i - \frac{m}{2}\right) - 1, & \text{if } \frac{m}{2} + 1 \leq i \leq m, \end{cases}$$

$$f(x'_i) = \begin{cases} m + 2i, & \text{if } 1 \leq i \leq \frac{m}{2}, \\ m + 2\left(i - \frac{m}{2}\right) - 1, & \text{if } \frac{m}{2} + 1 \leq i \leq m, \end{cases}$$

$$f(y_j) = \begin{cases} 2m + 2j, & \text{if } 1 \leq j \leq \frac{n}{2}, \\ 2m + 2\left(j - \frac{n}{2}\right) - 1, & \text{if } \frac{n}{2} + 1 \leq j \leq n, \end{cases}$$

$$f(y_j') = \begin{cases} 2m+n+2j, & \text{if } 1 \le j \le \frac{n}{2}, \\ 2m+n+2\left(j-\frac{n}{2}\right)-1, & \text{if } \frac{n}{2}+1 \le j \le n. \end{cases}$$

Case 2. m and n are odd.

$$f(x_i) = \begin{cases} 2i, & \text{if } 1 \le i \le \frac{m-1}{2}, \\ 2\left(i-\frac{m-1}{2}\right)-1, & \text{if } \frac{m-1}{2}+1 \le i \le m, \end{cases}$$

$$f(x_i') = \begin{cases} m+2i, & \text{if } 1 \le i \le \frac{m-1}{2}, \\ m+2\left(i-\frac{m-1}{2}\right)-1, & \text{if } \frac{m-1}{2}+1 \le i \le m, \end{cases}$$

$$f(y_j) = \begin{cases} 2m+2j, & \text{if } 1 \le j \le \frac{n-1}{2}, \\ 2m+2\left(j-\frac{n-1}{2}\right)-1, & \text{if } \frac{n-1}{2}+1 \le j \le n, \end{cases}$$

$$f(y_j') = \begin{cases} 2m+n+2j, & \text{if } 1 \le j \le \frac{n-1}{2}, \\ 2m+n+2\left(j-\frac{n-1}{2}\right)-1, & \text{if } \frac{n-1}{2}+1 \le j \le n. \end{cases}$$

Case 3. m is even and n is odd.

$$f(x_i) = \begin{cases} 2i, & \text{if } 1 \le i \le \frac{m}{2}, \\ 2\left(i-\frac{m}{2}\right)-1, & \text{if } \frac{m}{2}+1 \le i \le m, \end{cases}$$

$$f(x_i') = \begin{cases} m+2i, & \text{if } 1 \le i \le \frac{m}{2}, \\ m+2\left(i-\frac{m}{2}\right)-1, & \text{if } \frac{m}{2}+1 \le i \le m, \end{cases}$$

$$f(y_j) = \begin{cases} 2m+2j, & \text{if } 1 \le j \le \frac{n-1}{2}, \\ 2m+2\left(j-\frac{n-1}{2}\right)-1, & \text{if } \frac{n-1}{2}+1 \le j \le n, \end{cases}$$

$$f(y_j') = \begin{cases} 2m+n+2j, & \text{if } 1 \le j \le \frac{n-1}{2}, \\ 2m+n+2\left(j-\frac{n-1}{2}\right)-1, & \text{if } \frac{n-1}{2}+1 \le j \le n. \end{cases}$$

Case 4. m is odd and n is even.

$$f(x_i) = \begin{cases} 2i, & \text{if } 1 \le i \le \frac{m-1}{2}, \\ 2\left(i-\frac{m-1}{2}\right)-1, & \text{if } \frac{m-1}{2}+1 \le i \le m, \end{cases}$$

$$f(x_i') = \begin{cases} m+2i, & \text{if } 1 \le i \le \frac{m-1}{2}, \\ m+2\left(i-\frac{m-1}{2}\right)-1, & \text{if } \frac{m-1}{2}+1 \le i \le m, \end{cases}$$

$$f(y_j) = \begin{cases} 2m + 2j, & \text{if } 1 \leq j \leq \frac{n}{2}, \\ 2m + 2\left(j - \frac{n}{2}\right) - 1, & \text{if } \frac{n}{2} + 1 \leq j \leq n, \end{cases}$$

$$f(y'_j) = \begin{cases} 2m + n + 2j, & \text{if } 1 \leq j \leq \frac{n}{2}, \\ 2m + n + 2\left(j - \frac{n}{2}\right) - 1, & \text{if } \frac{n}{2} + 1 \leq j \leq n. \end{cases}$$

In Case 2, we obtain $e_f(1) = \frac{3mn-1}{2}$ and $e_f(0) = \frac{3mn+1}{2}$, while in all other cases $e_f(1) = e_f(0) = \frac{3mn}{2}$. Hence, $|e_f(0) - e_f(1)| \leq 1$. ∎

Example 2.9. Figure 4 illustrates a sum divisor cordial labeling of $S'(K_{2,2})$, where solid edges represent edges labeled 1 and dashed edges represent edges labeled 0.

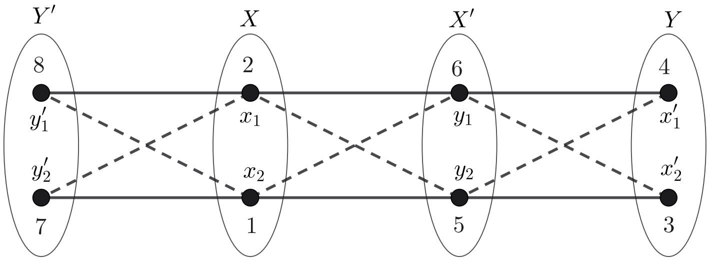

Figure 4: A sum divisor cordial labeling of $S'(K_{2,2})$

3 Concluding Remarks

In this work, we have studied sum divisor cordial labelings for several families of graphs, including the complete bipartite graph $K_{m,n}$, its splitting graph $S'(K_{m,n})$, the fan graph $F_{m,n}$, and the bi-usual fan graph $B(F_{1,n})$.

By constructing explicit labelings, we have shown that all these graphs admit sum divisor cordial labelings for all natural numbers m and n. These results extend previous findings on smaller graphs such as $F_{2,n}$, $K_{1,n}$, $K_{2,n}$, $K_2 + mK_1$, and $S'(K_{1,n})$. Our study thus provides a unified approach to sum divisor cordial labelings across several important classes of graphs and may serve as a foundation for further investigations on related graph families.

References

[1] D.G. Adalja, Sum divisor cordial labeling in the context of graphs operations on bistar, *TWMS Journal of Applied and Engineering Mathematics*, 12 (2), (2022), 588–599.

[2] D.G. Adalja and G.V. Ghodasara, Some new sum divisor cordial graphs, *International Journal of Applied Graph Theory*, 2, (2018), 19–33.

[3] D.G. Adalja, Sum divisor cordial labeling for vertex switching of special graphs, *International Journal of Applied Graph Theory*, 5 (2), (2021), 34–43.

[4] D. Adalja and P. Patel, Sum divisor cordial labeling for path union of two copies of cycle related graphs,*Palestine Journal of Mathematics*, 12 (4), (2023), 125–132.

[5] D.V. Annamma and P. Vijitha, Sum divisor cordial labeling of certain graphs, *Journal of Algebraic Statistics*, 13 (2), (2022), 3238–3242.

[6] J.A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 21, (2018), #DS6.

[7] A. Lourdusamy and F. Patrick, Sum divisor cordial graphs, *Proyecciones Journal of Mathematics*, 35 (1), (2016), 119–136.

[8] A. Lourdusamy and F. Patrick, Sum divisor cordial labeling for star and ladder related graphs, *Proyecciones Journal of Mathematics*, 35 (4), (2016), 437–455.

[9] M. Subbulakshmi and I. Valliammal, Decomposition of generalized fan graphs, *Advances in Mathematics: Scientific Journal*, 10 (5), (2021), 2381–2392.

[10] A. Sugumaran and K. Rajesh, Sum divisor cordial labeling on some special graphs, *International Journal of Mathematics and its Applications*, 6(2–B), (2018), 347–353.

[11] A. Sugumaran and K. Rajesh, Extended results on sum divisor cordial labeling, *Proyecciones Journal of Mathematics*, 38 (4), (2019), 653–663.